The unreasonable effectiveness of the tensor product.

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This paper is dedicated to Boris Venkov.

Abstract. Using the Hermitian tensor product description of the extremal even unimodular lattice of dimension 72 in [**15**] we show its extremality with the methods from [**4**].
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1. Introduction

The paper [**15**] describes the construction of an extremal even unimodular lattice Γ of dimension 72 of which the existence was a longstanding open problem. There are at least three independent proofs of extremality of this lattice, two of them are given in [**15**] and rely heavily on computations within the set of minimal vectors of the Leech lattice. The other one is also highly computational and uses the methods of [**16**]. All these proofs do not give much structural insight why this lattice is extremal. The present paper uses the construction of Γ as a Hermitian tensor product to derive a more structural proof of extremality of Γ with the methods in [**4**]. Moreover, the computational complexity of this new proof is far lower than the previously known ones.

Let L be a lattice in Euclidean ℓ-space $(\mathbb{R}^\ell, x\cdot y)$. Then the *dual lattice* is $L^* := \{x \in \mathbb{R}^\ell \mid x\cdot\lambda \in \mathbb{Z} \text{ for all } \lambda \in L\}$. The lattice is called *unimodular* (resp. *modular*), if L is equal (resp. similar to) L^*. Being (uni-)modular implies certain invariance properties of the theta series of L. In particular the theta series of an even unimodular lattice is a modular form for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. The theory of modular forms allows to show that the *minimum*

$$\min(L) := \min\{\lambda\cdot\lambda \mid 0 \neq \lambda \in \mathrm{L}\}$$

of L is bounded from above by $2 + 2\lfloor\frac{\ell}{24}\rfloor$. Lattices achieving equality are called *extremal*.

Several examples of extremal (uni-)modular lattices obtained as Hermitian tensor products of lower dimensional lattices were already known, see for instance [**1**] for a construction of extremal lattices of dimension 40 and 80 related to the Mathieu group M_{22}. This situation is nevertheless rather exceptional. Briefly, in order that a tensor product $L\otimes M$ gives rise to a dense sphere packing, it has to contain

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simultaneously *split* and *non split* short vectors. Obviously, the minimal length of a split vector $l \otimes m$ is exactly $\min L \min M$ while the minimal length of a non split vector $\sum_{i=1}^{r} l_i \otimes m_i$ $(r > 1)$ will usually be strictly smaller. The challenge, when allowing non split minimal vectors, is thus precisely to prevent their minimal length from dropping.

In the first section of this note, we review rather well-known results about the minima of tensor products of lattices over $\mathbb{Z}$, mainly due to Kitaoka. Also, and maybe less well-known, we comment on the behaviour of tensor product with respect to the associated sphere packing density. Roughly speaking, we show that the tensor product of two lattices over $\mathbb{Z}$ of small dimension cannot achieve a maximal density, even locally see Proposition 2.2 and its corollary (here "small" means "less than 43").

In contrast, tensor product over small field extensions, *e.g.* imaginary quadratic, may produce examples of dense or extremal lattices, among which the constructions already mentioned, in particular the extremal lattice Γ in dimension 72. Section 3 recalls some facts on Hermitian lattices over imaginary quadratic number fields. These are then applied to give a construction of one extremal even unimodular 48-dimensional lattices as a Hermitian tensor product over $\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$ in Section 4 before we give a new proof of the extremality of Γ in Section 5.

2. Tensor products over $\mathbb{Z}$

In this section, we analyze the behaviour of tensor product of Euclidean lattices with respect to *perfection*, a notion which we first recall.

Let L be a Euclidean lattice equipped with a basis $\mathcal{B}$. We denote by $S(L)$ the set of its minimal vectors (non zero vectors of shortest length). To every minimal vector x we associate the integral column vector X of its coordinates on $\mathcal{B}$, and denote $S_{\mathcal{B}}$ the set of such Xs as x runs through $S(L)$. The *rank of perfection* of L is the integer

$$r_{perf}(L) = \dim \operatorname{Span}_{\mathbb{R}} \left\{ XX^t \ \mid \ X \in S_{\mathcal{B}}(L) \right\}.$$

Clearly, $r_{perf}(L)$ does not depend on the choice of a particular basis $\mathcal{B}$, and is at most $\dfrac{\ell(\ell+1)}{2}$, where $\ell = \operatorname{rank} L$, since XX^t is a symmetric matrix of size ℓ for all $X \in S_{\mathcal{B}}(L)$.

Definition 2.1. A lattice L of rank ℓ is perfect if $r_{perf}(L) = \dfrac{\ell(\ell+1)}{2}$.

Lattices achieving a local maximum of density are classically called *extreme*. Perfection is a necessary condition for a lattice to be extreme, as was first observed by Korkine and Zolotareff (see [**11**, Chapter 3] for historical comments).

Every element of the tensor product $L \otimes_{\mathbb{Z}} M$ of two Euclidean lattices can be written as a sum of *split* vectors $x \otimes y$ $(x \in L,\ y \in M)$. The Euclidean structure on $L \otimes_{\mathbb{Z}} M$ is defined, on split vectors, by the formula

$$(x \otimes y) \cdot (z \otimes t) = (x \cdot z)\,(y \cdot t)$$

which extends uniquely to a well-defined inner product on $L \otimes_{\mathbb{Z}} M$.

Proposition 2.2. *Let L and M be Euclidean lattices of rank at least 2. If all the minimal vectors of $L \otimes_{\mathbb{Z}} M$ are split, then $L \otimes M$ is not perfect, and consequently not extreme.*

PROOF. Fix bases $\mathcal{B}$ and $\mathcal{C}$ of L and M respectively. Under the hypothesis that all minimal vectors of $L \otimes_{\mathbb{Z}} M$ are split we have

$$\begin{aligned} r_{perf}(L \otimes_{\mathbb{Z}} M) &= \dim \operatorname{Span}_{\mathbb{R}} \left\{ (X \otimes Y)(X \otimes Y)^t \ \mid \ X \in S_{\mathcal{B}}(L),\ Y \in S_{\mathcal{C}}(M) \right\} \\ &= \dim \operatorname{Span}_{\mathbb{R}} \left\{ XX^t \otimes YY^t \ \mid \ X \in S_{\mathcal{B}}(L),\ Y \in S_{\mathcal{C}}(M) \right\} \\ &\leq \frac{\ell(\ell+1)}{2} \frac{m(m+1)}{2} \\ &< \frac{\ell m(\ell m+1)}{2} \end{aligned}$$

whence the conclusion. □

The question as to whether the minimal vectors of a tensor product are split or not has been investigated thoroughly by Kitaoka (see [**9**, Chapter 7]). Combining some of his results with the previous proposition one obtains :

COROLLARY 2.3. *If* $\operatorname{rank} L \leq 43$ *or* $\operatorname{rank} M \leq 43$*, then* $L \otimes M$ *is not perfect, and consequently not extreme.*

PROOF. By [**9**, Theorem 7.1.1] we know that if the conditions of the corollary are satisfied, then the minimal vectors are split, whence the conclusion using Proposition 2.2 □

REMARK 2.4.
(1) To our knowledge, no explicit examples of lattices L and M such that $L \otimes_{\mathbb{Z}} M$ contains non split minimal vectors is known (it would require L and M to have rank at least 44). However, it is known thanks to an unpublished theorem of Steinberg (see [**8**, Theorem 9.6]) that in any dimension $n \geq 292$ there exist unimodular lattices L and M such that $\min L \otimes_{\mathbb{Z}} M < \min L \min M$ (the proof is of course non constructive).
(2) As is well-known, extremal even unimodular lattices of dimension $24k$ or $24k+8$ are extreme (cf. [**2**] also for the modular analogues), hence perfect. Consequently, there is no hope to obtain new extremal modular lattices in dimension $24k$ or $24k+8 \leq 43^2$ as tensor product *over* $\mathbb{Z}$ of lattices in smaller dimensions. Note that this also follows from the definition of extremality since for $\ell, m \geq 8$

$$(2+2\lfloor \frac{\ell}{24} \rfloor)(2+2\lfloor \frac{m}{24} \rfloor) < (2+2\lfloor \frac{\ell m}{24} \rfloor).$$

3. Preliminaries on Hermitian lattices

For sake of completeness, we recall in this section some basic notation and lemmas about Hermitian lattices (see [**4**] or [**7**] for complete proofs). Let K be an imaginary quadratic field, with ring of integers $\mathcal{O}_K$. The non trivial Galois-automorphism of K is denoted by $\bar{\ }$ (identified with the classical complex conjugation if an embedding of K in $\mathbb{C}$ is fixed). We denote by $\mathcal{D}_{K/\mathbb{Q}}$ the different of $K/\mathbb{Q}$ and $\mathfrak{d}_K$ its discriminant. A Hermitian lattice in a finite-dimensional K-vector space V, endowed with a positive definite Hermitian form h, is a finitely generated $\mathcal{O}_K$-submodule of V containing a K-basis of V. The (Hermitian) dual of a Hermitian lattice L is defined as

$$L^{\#} = \{ y \in V \ \mid \ h(y, L) \subset \mathcal{O}_K \} .$$

Its *discriminant* d_L is defined via the choice of a pseudo-basis: writing $L = \mathfrak{a}_1 e_1 \oplus \cdots \oplus \mathfrak{a}_m e_m$, where $\{e_1, \ldots, e_m\}$ is a K-basis of $V \simeq K^m$ and the $\mathfrak{a}_i$s are fractional ideals in K, we define d_L as the unique positive generator in $\mathbb{Q}$ of the ideal $\det(h(e_i, e_j)) \prod \mathfrak{a}_i \overline{\mathfrak{a}_i}$. This definition is independent of the choice of a pseudo-basis $(\mathfrak{a}_i, e_i)$ and in the specific case where $\mathcal{O}_K$ is principal, one may take $\mathfrak{a}_i = \mathcal{O}_K$ for all i, and d_L is nothing but the determinant of the Hermitian Gram matrix of a basis of L.

As in the Euclidean case (see [**11**, Proposition 1.2.9]) we obtain the following lemma.

Lemma 3.1. *Let L be a Hermitian lattice, F a K-subspace of $KL = V$, p the orthogonal projection onto $F^\perp$. Then*

$$d_L = d_{F \cap L} d_{p(L)} \tag{3.1}$$

For any $1 \leq r \leq m = \operatorname{rank}_{\mathcal{O}_K} L$ we define $d_r(L)$ as the minimal discriminant of a free $\mathcal{O}_K$-sublattice of rank r of L. In particular, one has

$$d_1(L) = \min(L) := \min\{h(v, v) \mid 0 \neq v \in L\}.$$

The minimal discriminants of L and $L^\#$ satisfy the following symmetry relation, the proof of which is the same as in the Euclidean case (see [**11**, Proposition 2.8.4]).

Lemma 3.2. *Let L be a Hermitian lattice of rank m. Then, for any $1 \leq r \leq m-1$, one has*

$$d_L = d_r(L) d_{m-r}(L^\#)^{-1}. \tag{3.2}$$

By restriction of scalars, an $\mathcal{O}_K$-lattice of rank m can be viewed as a $\mathbb{Z}$-lattice of rank $2m$, the *trace lattice of* L, with inner product defined by

$$x \cdot y = \operatorname{Tr}_{K/\mathbb{Q}} h(x, y). \tag{3.3}$$

The dual L^* of L with respect to that inner product is linked to $L^\#$ by

$$L^* = \mathcal{D}_{K/\mathbb{Q}}^{-1} L^\# \tag{3.4}$$

whence the relation

$$\det L = |\mathfrak{d}_K|^m (d_L)^2. \tag{3.5}$$

Note that, because of (3.3), the minimum of L, viewed as an ordinary $\mathbb{Z}$-lattice, is twice its "Hermitian" minimum $d_1(L)$. To avoid any confusion, we stick to Hermitian minima in what follows.

For the proof of the main result, we use the technique developed in [**4**] to bound the minimum of a Hermitian tensor product. Suppose L and M are Hermitian lattices over a number field K. Then any vector $z \in L \otimes_{\mathcal{O}_K} M$ is a sum of tensors of the form $v \otimes w$ with $v \in L$ and $w \in M$. The minimal number of summands in such an expression is called the *rank* of z. Clearly the rank of any vector is less than the minimum of the dimension of the two tensor factors.

As in the Euclidean case, the Hermitian structure on $L \otimes_{\mathcal{O}_K} M$ is defined, on split vectors, by the formula

$$h(x \otimes y, z \otimes t) = h(x, z)\, h(y, t)$$

which extends uniquely to a well-defined positive definite Hermitian form on $L \otimes_{\mathcal{O}_K} M$.

PROPOSITION 3.3. *([**4**, Proposition 3.2]) Let L and M be Hermitian lattices and denote by $d_r(L)$ the minimal determinant of a rank r sublattice of L. Then for any vector $z \in L \otimes_{\mathcal{O}_K} M$ of rank r one has*

$$h(z,z) \geq r d_r(L)^{1/r} d_r(M)^{1/r}. \tag{3.6}$$

Moreover, a vector z of rank r in $L \otimes_{\mathcal{O}_K} M$ for which equality holds in (3.6) exists if and only if M and L contain minimal r-sections M_r and L_r such that $M_r \simeq L_r^{\#}$.

PROOF. The inequality (3.6) is precisely [**4**, Proposition 3.2]. The last assertion follows from close inspection of the proof, which shows that $h(z,z) = r d_r(L)^{1/r} d_r(M)^{1/r}$ if and only if $z = \sum_{i=1}^{r} e_i \otimes f_i$ where $\{e_1, \ldots, e_r\}$, resp. $\{f_1, \ldots, f_r\}$, are $\mathcal{O}_K$-bases of minimal sections M_r and L_r of M and L respectively, such that $(h(e_i,e_j))_{i,j} = \overline{(h(f_i,f_j))_{i,j}}^{-1}$. □

3.1. Two dimensional Hermitian lattices. The results in this section are certainly well known, we include them together with the short proof for completeness.

DEFINITION 3.4. The *Euclidean minimum* of $\mathcal{O}_K$ is defined as

$$\mu(\mathcal{O}_K) := \sup_{x \in K} \inf_{a \in \mathcal{O}_K} N_{K/\mathbb{Q}}(x-a).$$

An element $z \in K$ such that $N(z) = \inf_{a \in \mathcal{O}_K} N_{K/\mathbb{Q}}(z-a) = \mu(\mathcal{O}_K)$ is called a *deep hole* of O_K.

Note that the Euclidean minimum is just the covering radius of the lattice $\mathcal{O}_K$ with respect to the positive definite bilinear form $x \cdot y := \frac{1}{2} \operatorname{Tr}_{K/\mathbb{Q}}(x\overline{y})$. Also, $\mathcal{O}_K$ is a Euclidean ring if $\mu(\mathcal{O}_K) < 1$.

PROPOSITION 3.5. *Assume that $\mu := \mu(\mathcal{O}_K) < 1$ and let L be a 2-dimensional Hermitian $\mathcal{O}_K$-lattice with $\min(L) = m$. Then $d_L \geq m^2(1-\mu)$.*

PROOF. The proof follows the argument of [**4**, Lemma 4.2.2]. Let $x \in L$ be a minimal vector of L and extend it to an $\mathcal{O}_K$-basis of $L = \mathcal{O}_K x + \mathcal{O}_K y$. Let $p(y) = bx$ denote the projection of y onto $\langle x \rangle$. Replacing y by $y - ax$ with $a \in \mathcal{O}_K$ such that $N_{K/\mathbb{Q}}(a-b) \leq \mu$ we may assume that $N_{K/\mathbb{Q}}(b) = b\overline{b} \leq \mu$. Then

$$\begin{array}{lll} d_L & = h(x,x)h(y-p(y),y-p(y)) & \geq h(x,x)(h(y,y) - \mu h(x,x)) \\ & \geq (1-\mu)h(x,x)h(y,y) & \geq (1-\mu)m^2. \end{array}$$

□

REMARK 3.6. The proof shows that for $\mu < 1$ any 2-dimensional lattice L has an $\mathcal{O}_K$-basis (x,y) such that

$$h(x,x)h(y,y)(1-\mu) \leq d_L.$$

The norm Euclidean imaginary quadratic number fields $\mathbb{Q}[\sqrt{-d}]$. The last two lines give the orbit representatives of the deep holes under the action of $(\mathcal{O}_K^*) : \langle \overline{} \rangle$

d	3	1	7	2	11
μ	1/3	1/2	4/7	3/4	9/11
$(1-\mu)d_K$	2	2	3	2	2
# deep holes	6	4	6	4	6
orbit repr. of deep holes	$\frac{1}{\sqrt{-3}}$	$\frac{1}{1+i}$	$2/\sqrt{-7}$ $\frac{7+3\sqrt{-7}}{14}$	$\frac{1+\sqrt{-2}}{2}$	$3/\sqrt{-11}$ $\frac{11+5\sqrt{-11}}{22}$

Corollary 3.7. *Let $z \in K$ be a deep hole of $\mathcal{O}_K$. Then the lattice L_K with Gram matrix $\begin{pmatrix} 1 & z \\ \overline{z} & 1 \end{pmatrix}$ the unique (up to $\mathcal{O}_K$-linear or antilinear isometry) densest 2-dimensional Hermitian $\mathcal{O}_K$-lattice. The 4-dimensional $\mathbb{Z}$-lattice $(L_K, \mathrm{Tr}_{K/\mathbb{Q}}(h))$ is isometric to the root lattice D_4 for $d = 3, 1, 2, 11$ and to $A_2 \perp A_2$ for $d = 7$.*

This might give some hint of why tensor products of Hermitian lattices over $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$ seem to be more successful than over other rings of integers in imaginary quadratic fields.

Also note that for $d = 7$ and $d = 11$, where there are 2 orbits of deep holes, the corresponding lattices L_K are isometric.

4. Hermitian $\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$-lattices.

We now apply the theory from above to the special case $K = \mathbb{Q}[\sqrt{-11}]$. Let $\eta := \frac{1+\sqrt{-11}}{2}$. Then $\eta^2 - \eta + 3 = 0$ and $\mathcal{O}_K = \mathbb{Z}[\eta]$ is an Euclidean domain with Euclidean minimum $\frac{9}{11}$.

The Hermitian $\mathcal{O}_K$-structures of the Leech lattice have not been classified. However we may construct some of them using the classification of finite quaternionic matrix groups in [**14**] and embeddings of K into definite quaternion algebras. It turns out that we obtain three different $\mathcal{O}_K$-structures, P_1, P_2 and P_3, with automorphism groups $\mathrm{Aut}_{\mathcal{O}_K}(P_1) \cong 2.G_2(4)$ (with endomorphism algebra $\mathcal{Q}_{\infty,2}$), $\mathrm{Aut}_{\mathcal{O}_K}(P_2) \cong (L_2(7) \times \tilde{S}_3).2$ (with endomorphism algebra $\mathcal{Q}_{\infty,7}$), and $\mathrm{Aut}_{\mathcal{O}_K}(P_3) \cong \mathrm{SL}_2(13).2$ (with endomorphism algebra $\mathcal{Q}_{\infty,13}$).

Proposition 4.1. *Let T be the 2-dimensional unimodular Hermitian $\mathcal{O}_K$-lattice with Gram matrix $\begin{pmatrix} 2 & \eta \\ \overline{\eta} & 2 \end{pmatrix}$. Let (P, h) be some 12-dimensional $\mathcal{O}_K$ lattice such that the trace lattice $(P, \mathrm{Tr}_{K/\mathbb{Q}} \circ h)$ is isometric to the Leech lattice. Then the Hermitian tensor product $R := P \otimes_{\mathcal{O}_K} T$ has minimum either 2 or 3. The minimum of R is 3, if and only if (P, h) does not represent one of the lattices L_K or T.*

Proof. The trace lattice of R is an even unimodular lattice of dimension 48, so the Hermitian minimum of R is either 1, 2, or 3 and for any $v \in R$ we have $h(v, v) \in \mathbb{Z}$. So let $0 \neq v \in R$. In order to apply Proposition 3.3 we need to deal with the two cases that the rank of v is 1 or 2. If the rank of v is 1, then $v = p \otimes t$ is a pure tensor and $h(v, v) \geq \min(P)\min(T) = 4$. If the rank of v is 2, then by Proposition 3.3

$$h(v, v) \geq 2 d_2(P)^{1/2}, \text{ because } d_2(T) = d_T = 1.$$

Since $d_2(P) \geq 2^2(1 - \mu) = \frac{8}{11}$ the norm $h(v, v) \geq 2$ and $h(v, v)$ is strictly bigger than 2, if $d_2(P) > 1$. So let $L \leq P$ be a 2-dimensional sublattice of determinant $d_L \leq 1$. By Remark 3.6 the lattice L has a basis (x, y) such that

$$(1 - \mu)h(x, x)h(y, y) = \frac{2}{11} h(x, x)h(y, y) \leq d_L \leq 1.$$

This implies that $h(x, x) = h(y, y) = 2$ and the Gram matrix of (x, y) is

$$\begin{pmatrix} 2 & z \\ \overline{z} & 2 \end{pmatrix}$$

for some $z \in \frac{1}{\sqrt{-11}}\mathcal{O}_K$. Since the minimum of L is 2 and the densest 2-dimensional $\mathcal{O}_K$-lattice of minimum 2 has determinant $\frac{8}{11}$ we obtain

$$4 - z\overline{z} \in \{\frac{8}{11}, \frac{9}{11}, \frac{10}{11}, 1\}$$

There are no elements in K with norm $\frac{35}{11}$ or $\frac{34}{11}$, so the middle two possibilities are excluded. For the other two lattices we find $N(z) = z\overline{z} = 3$ and then $L \cong T$ or $N(z) = \frac{36}{11}$ and $L \cong L_K$. $\square$

COROLLARY 4.2. *$\min(P_1 \otimes_{\mathcal{O}_K} T) = 2$ with kissing number $2 \cdot 196560$, $\min(P_2 \otimes_{\mathcal{O}_K} T) = 2$ with kissing number $2 \cdot 15120$, and $\min(P_3 \otimes_{\mathcal{O}_K} T) = 3$. The trace lattice of the latter is isometric to the extremal even unimodular lattice P_{48n} discovered in* [**13**].

PROOF. For $P = P_1$, P_2, and P_3 we computed orbit representatives of the $\mathrm{Aut}_{\mathcal{O}_K}(P)$-action on the set S of minimal vectors of P. For each orbit representative v we computed all inner products $h(v, w)$ with $w \in S$ to obtain the representation number of T and L_K by P.
Let $P = P_1$. Then $\mathcal{M} = \mathrm{End}_{\mathrm{Aut}_{\mathcal{O}_K}(P)}(P)$ is the maximal order in the quaternion algebra $\mathcal{Q}_{\infty,2}$. Given $v \in S$ there is a unique sublattice

$$\langle v \rangle_{\mathcal{M}} = \langle v, w \rangle_{\mathcal{O}_K} \cong_{\mathcal{O}_K} L_K.$$

The lattice P_1 does not represent the lattice T. The lattice P_2 represents both lattices, T and L_K, with multiplicity 10080 and 5040 respectively. Only the lattice P_3 represents neither T nor L_K. $\square$

5. Hermitian $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$-lattices.

We now restrict to the special case $K = \mathbb{Q}[\sqrt{-7}]$. Then $\mathcal{O}_K = \mathbb{Z}[\alpha]$ where $\alpha^2 - \alpha + 2 = 0$. Put $\beta := \overline{\alpha} = 1 - \alpha$ its complex conjugate. Then $\mathbb{Z}[\alpha]$ is an Euclidean domain with Euclidean minimum $\frac{4}{7}$.

Let (P, h) be a Hermitian $\mathbb{Z}[\alpha]$-lattice, so P is a free $\mathbb{Z}[\alpha]$-module and $h : P \times P \to \mathbb{Q}[\alpha]$ a positive definite Hermitian form. One example of such a lattice is the *Barnes lattice* P_b with Hermitian Gram matrix

$$\begin{pmatrix} 2 & \alpha & -1 \\ \beta & 2 & \alpha \\ -1 & \beta & 2 \end{pmatrix}$$

Then P_b is Hermitian unimodular, $P_b = P_b^{\#}$ and has Hermitian minimum $\min(P_b) = 2$.

We will make use of the following two facts:

Fact 1:

(a) $d_1(P_b) = 2$.
(b) $d_2(P_b) = 2$.
(c) $d_3(P_b) = d_{P_b} = 1$.

Fact 2:

(a) By Proposition 3.5 the unique densest 2-dimensional $\mathbb{Z}[\alpha]$-lattice is the lattice P_a with Gram matrix $\begin{pmatrix} 2 & 4/\sqrt{-7} \\ -4/\sqrt{-7} & 2 \end{pmatrix}$, $\min(P_a) = 2$, and $d_{P_a} = 12/7$.

(b) There is a version of Voronoi theory also for Hermitian lattices developed in [**5**]. This is used in the thesis [**12**] to classify the densest $\mathbb{Z}[\alpha]$-lattices in dimension 3. From this it follows that P_b is the globally densest 3-dimensional Hermitian $\mathbb{Z}[\alpha]$-lattice.

REMARK 5.1. The densest 8-dimensional $\mathbb{Z}$-lattice E_8 has a structure as a Hermitian $\mathbb{Z}[\alpha]$-lattice P_c of dimension 4, which therefore realises the unique densest 4-dimensional $\mathbb{Z}[\alpha]$-lattice.

From the two facts above we immediately obtain the following Proposition.

PROPOSITION 5.2. *Let* (P, h) *be a Hermitian* $\mathbb{Z}[\alpha]$ *lattice of dimension* ≥ 3 *and with* $\min(P) =: m$. *Then*

(a) $d_1(P) = \min(P) = m$.
(b) $d_2(P) \geq \frac{3m^2}{7}$.
(c) $d_3(P) \geq \frac{m^3}{8}$ *and* $d_3(P) = \frac{m^3}{8}$ *if and only if* P *contains a sublattice isometric to* $\sqrt{m/2}P_b$.

5.1. An application to unimodular 72-dimensional lattices. We now apply the theory from the previous sections to obtain a new proof for the extremality of the even unimodular lattice Γ in dimension 72 from [**15**]. Michael Hentschel [**6**] classified all Hermitian $\mathbb{Z}[\alpha]$-structures on the even unimodular $\mathbb{Z}$-lattices of dimension 24 using the Kneser neighbouring method [**10**] to generate the lattices and checking completeness with the mass formula. In particular there are exactly nine such $\mathbb{Z}[\alpha]$ structures (P_i, h) $(1 \leq i \leq 9)$ such that the trace lattice $(P_i, \mathrm{Tr}_{\mathbb{Z}[\alpha]/\mathbb{Z}} \circ h) \cong \Lambda$ is the Leech lattice. They are used by the second author in [**15**] to construct nine 36-dimensional Hermitian $\mathbb{Z}[\alpha]$-lattice R_i defined by $(R_i, h) := P_b \otimes_{\mathbb{Z}[\alpha]} P_i$. Using the methods described above we obtain the following main result on the minimum of these tensor products.

THEOREM 5.3. *The minimum of the Hermitian lattices* R_i *is either* 3 *or* 4. *The number of vectors of norm* 3 *in* R_i *is equal to the representation number of* P_i *for the sublattice* P_b. *In particular* $\min(R_i) = 4$ *if and only if the Hermitian Leech lattice* P_i *does not contain a sublattice isomorphic to* P_b.

PROOF. The proof follows from Proposition 3.3 and uses Proposition 5.2: (An alternative proof that is not based on the computation of perfect $\mathbb{Z}[\alpha]$-lattices is given in the next section.)
Let $z \in P_i \otimes_{\mathbb{Z}[\alpha]} P_b$ be a non-zero vector of rank $r = 1, 2$, or 3.

- If $r = 1$, then $z = v \otimes w$ and $h(z, z) \geq \min(P_i)\min(P_b) = 4$.
- If $r = 2$, then $h(z, z) \geq 2\sqrt{d_2(P_b)}\sqrt{d_2(P_i)} \geq 2\sqrt{2}\sqrt{\frac{12}{7}} > 3$, so $h(z, z) \geq 4$.
- If $r = 3$, then $h(z, z) \geq 3d_3(P_i)^{1/3} \geq 3$. Since $h(z, z) \in \mathbb{Z}$ this implies that $h(z, z) \geq 3$ and equality requires that P_i contains a minimal section isometric to $P_b^{\#} = P_b$.

□

COROLLARY 5.4. *Let* P_1 *denote the Hermitian Leech lattice with automorphism group* $\mathrm{SL}_2(25)$ *(see* [**15**]*). Then* $\min(P_1 \otimes_{\mathbb{Z}[\alpha]} P_b) = 4$. *For the other eight Hermitian Leech lattices* P_i *the minimum is* $\min(P_i \otimes_{\mathbb{Z}[\alpha]} P_b) = 3$ *(*$i = 2, \ldots, 9$*).*

PROOF. With MAGMA ([**3**]) we computed the number of sublattices isomorphic to P_b in the lattices P_i. Only one of them, P_1, does not contain such a sublattice, so $d_3(P_1) > 1$ and hence $\min(P_1 \otimes_{\mathbb{Z}[\alpha]} P_b) \geq 4$. For the computation we went through orbit representatives v_1 of the Hermitian automorphism group $\mathrm{Aut}(P_i)$ on the set S of minimal vectors of the Leech lattice. For any v_1 we compute the set

$$A(v_1) := \{v \in S \mid h(v, v_1) = \alpha\}.$$

In all cases this set $A(v_1)$ has 32 elements. For all $v_2 \in A(v_1)$ we count the number of vectors $v \in S$ such that $h(v, v_2) = \alpha$ and $h(v, v_1) = -1$. This computation takes about 30 seconds per orbit representative v_1. □

5.2. An alternative proof of Theorem 5.3. The thesis [**12**] uses the Voronoi algorithm to compute the 3-dimensional perfect $\mathbb{Z}[\alpha]$-lattices. The proof of Theorem 5.3 only uses the following proposition which can be proved without computer.

PROPOSITION 5.5. *Let P be one of the nine $\mathbb{Z}[\alpha]$-structures of the Leech lattices Λ_{24}. Then*

(a) $d_1(P) = \min(P) = 2$.
(b) $d_2(P) = \frac{12}{7}$.
(c) $d_3(P) \geq 1$.

PROOF. (a) follows from the fact that the Leech lattice is extremal.
(b) By Proposition 3.5 the discriminant d_M of a $\mathbb{Z}[\alpha]$-lattice M of rank 2 satisfies

$$d_M \geq \frac{3}{7}\min(M)^2.$$

If M is a sublattice of P, then $\min(M) \geq 2$ and hence $d_M \geq \frac{12}{7}$. On the other hand all nine Hermitian structures contain sublattices P_a of determinant $\frac{12}{7}$.
(c) Assume by way of contradiction that $d_3(P) < 1$. Since $P^{\#} = \sqrt{-7}P$, we have $h(x, y) \in \frac{1}{\sqrt{-7}}\mathbb{Z}[\alpha]$ for any x, y in P, and moreover, since P is even as a Euclidean lattice, we see that $h(x, x) \in \mathbb{Z}$ for $x \in P$. As a consequence, if $M = \oplus_{i=1}^{3}\mathbb{Z}[\alpha]e_i$ is a 3-dimensional section of P, its discriminant $d_M\colon = \det(h(e_i, e_j))$ belongs to $\frac{1}{7}\mathbb{Z}$. In particular

$$d_M < 1 \Longrightarrow d_M \leq \frac{6}{7}.$$

Furthermore, $\gamma_h(M)\colon = \dfrac{\min M}{d_M^{1/3}}$ is bounded from above (see [**4**]) by

$$\gamma_h(M) \leq \frac{\sqrt{7}}{2}\gamma_6 = \frac{\sqrt{7}}{3^{1/6}} \simeq 2.203 \tag{5.1}$$

which immediately implies that $d_M \geq \dfrac{8\sqrt{3}}{7\sqrt{7}} > 5/7$. We conclude that

$$d_M < 1 \Longrightarrow d_M = \frac{6}{7}.$$

Next we show that if such a sublattice M with $d_M = \dfrac{6}{7}$ exists, then it admits a minimal 2-dimensional subsection generated over $\mathbb{Z}[\alpha]$ by two minimal vectors of

P. Otherwise we would have, by Remark 3.6,

$$d_2(M) \geq \frac{3}{7} 3 \cdot 2 = \frac{18}{7}$$

whence, using the identity $d_M = d_2(M) d_1(M^\#)^{-1}$ (see Lemma 3.2),

$$\gamma_h(M^\#) \geq \frac{d_2(M)}{d_M^{2/3}} \geq 3 \left(\frac{6}{7}\right)^{1/3} \simeq 2.85$$

violating bound (5.1).

Thus, one can find a $\mathbb{Z}[\alpha]$-basis $\{e_1, e_2, e_3\}$ of M, such that $h(e_1, e_1) = h(e_2, e_2) = 2$ and $M_2 := \mathbb{Z}[\alpha]e_1 \oplus \mathbb{Z}[\alpha]e_2$ is a minimal 2-dimensional section of M. Setting $h(e_1, e_2) = \dfrac{a}{\sqrt{-7}}$, with $a \in \mathbb{Z}[\alpha]$ we see that

$$\frac{12}{7} = d_2(P) \leq \det \begin{pmatrix} 2 & \dfrac{a}{\sqrt{-7}} \\ -\dfrac{\overline{a}}{\sqrt{-7}} & 2 \end{pmatrix} = d_2(M) \leq \gamma_h(M^\#) d_M^{2/3} \leq \frac{\sqrt{7}}{3^{1/6}} \left(\frac{6}{7}\right)^{2/3} \simeq 1.988$$

which yields $14 < a\overline{a} \leq 16$, whence $a\overline{a} = 16$ (15 is not a norm), and $d_2(M) = d_2(P) = \dfrac{12}{7}$. Replacing e_2 by $\pm e_2 \pm e_1$ if necessary, we may finally assume that $h(e_1, e_2) = \dfrac{4}{\sqrt{-7}}$. Finally, we have the formula

$$\frac{6}{7} = d_M = d_{M_2} h\left(q(e_3), q(e_3)\right) = d_{M_2}\left(h(e_3, e_3) - h\left(p(e_3), p(e_3)\right)\right)$$

where p and q stand respectively for the orthogonal projection on the subspace $F := \mathbb{Q}[\alpha]M_2 = \mathbb{Q}[\alpha]e_1 + \mathbb{Q}[\alpha]e_2$ and its orthogonal complement $F^\perp$ (see Lemma 3.1). Furthermore, we may replace e_3 by $e_3 + u$, with $u \in M_2$, and it is easily seen that u may be chosen so that $h(p(e_3 + u), p(e_3 + u)) \leq \dfrac{80}{49}$ (the Hermitian norm of any vector $v = xe_1 + ye_2$ in F is given by $h(v, v) = \dfrac{2}{7}\left(7|x + \dfrac{2}{\sqrt{-7}} y|^2 + 3|y|^2\right)$, and since $\mathbb{Z}[\alpha]$ is Euclidean with Euclidean minimum $\dfrac{4}{7}$ we may choose y' and x' in $\mathbb{Z}[\alpha]$ such that $|y - y'|^2 \leq \dfrac{4}{7}$ and $|(x - x') + \dfrac{2}{\sqrt{-7}}(y - y')|^2 \leq \dfrac{4}{7}$, whence the conclusion). Consequently, one has

$$\frac{6}{7} = d_M \geq \frac{12}{7}\left(h(e_3, e_3) - \frac{80}{49}\right)$$

which implies that $h(e_3, e_3) = 2$.

Finally, the Hermitian Gram matrix of M is

$$\begin{pmatrix} 2 & 4/\sqrt{-7} & a/\sqrt{-7} \\ -4/\sqrt{-7} & 2 & b/\sqrt{-7} \\ -\overline{a}/\sqrt{-7} & -\overline{b}\sqrt{-7} & 2 \end{pmatrix}$$

with a, b in $\mathbb{Z}[\alpha]$, of norm at most 16 (this is because the determinant of any 2-dimensional section is at least $12/7$). Consequently, there are finitely many possible a and b, and it is not hard to check that, up to permutation of e_1 and e_2 and sign change for e_3, the only choice to achieve the condition $d_M = 6/7$ is $a = 3/\sqrt{-7}$ and

$b = 0$. But this leads to a Hermitian Gram matrix $\begin{pmatrix} 2 & 4/\sqrt{-7} & 3/\sqrt{-7} \\ -4/\sqrt{-7} & 2 & 0 \\ -3/\sqrt{-7} & 0 & 2 \end{pmatrix}$ of minimum 1, a contradiction. $\square$

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